

Grobman-Hartman Theorems for Diffeomorphisms of Banach Spaces over Valued Fields

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Abstract

Consider a local diffeomorphism f of an ultrametric Banach space over an ultrametric field, around a hyperbolic fixed point x . We show that, locally, the system is topologically conjugate to the linearized system. An analogous result is obtained for local diffeomorphisms of real p -Banach spaces (like ℓ^p), for $p \in]0, 1]$. More generally, we obtain a local linearization if f is merely a local homeomorphism which is strictly differentiable at a hyperbolic fixed point x . Also a new global version of the Grobman-Hartman theorem is provided. It applies to Lipschitz perturbations of hyperbolic automorphisms of Banach spaces over valued fields. The local conjugacies H constructed are not only homeomorphisms, but both H and H^{-1} are Hölder. In the global case, we also study the dependence of H and H^{-1} on f (keeping x and $f'(x)$ fixed).

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1 Introduction and statement of main results

The linearization problem for formal or analytic diffeomorphisms of a complete ultrametric field \mathbb{K} (or \mathbb{K}^n), via formal or analytic conjugacies, has

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attracted interest in non-archimedean analysis (see [17], [22] and [32]). Since an analytic linearization is not always possible, it is natural to ask whether at least a (local) topological conjugacy from the given system to its linearized version is available. In the current article, we answer this question in the affirmative (under natural hyperbolicity hypotheses).

More generally, for some of our results we can work with a valued field $(\mathbb{K}, |\cdot|)$ whose absolute value $|\cdot|$ is assumed to define a non-discrete topology on \mathbb{K} (such a field is called ultrametric if $|x+y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{K}$). If E is a Banach space over \mathbb{K} , we shall say that an automorphism $A: E \rightarrow E$ of topological vector spaces is *hyperbolic* if there exist A -invariant vector subspaces E_s and E_u of E such that $E = E_s \oplus E_u$, and a norm $\|\cdot\|$ on E defining its topology, such that

$$\|x + y\| = \max\{\|x\|, \|y\|\} \quad \text{for all } x \in E_s \text{ and } y \in E_u \quad (1)$$

and

$$\|A|_{E_s}\| < 1 \quad \text{and} \quad \|A^{-1}|_{E_u}\| < 1 \quad (2)$$

holds for the operator norms with respect to $\|\cdot\|$ (then call $\|\cdot\|$ *adapted to A*).

Our two main theorems are versions of the local and global Grobman-Hartman theorem for C^1 -diffeomorphisms of \mathbb{R}^n (see [12], [13], [15], [16], [19], [23] and [24] for these classical results and their analogues for flows). Our presentation is particularly indebted to [24]. We first discuss global conjugacies:

Theorem A (Global Grobman-Hartman Theorem) *Let E be a Banach space over a valued field $(\mathbb{K}, |\cdot|)$ and $A: E \rightarrow E$ be an automorphism of topological vector spaces which is hyperbolic. Let $\|\cdot\|: E \rightarrow [0, \infty[$ be a norm adapted to A and $g: E \rightarrow E$ be a bounded Lipschitz map such that*

$$\text{Lip}(g) < \|A^{-1}\|^{-1}, \quad \|A^{-1}|_{E_u}\|(1 + \text{Lip}(g)) < 1, \quad \text{and} \quad \|A|_{E_s}\| + \text{Lip}(g) < 1.$$

Then there exists a unique bounded continuous map $v: E \rightarrow E$ such that

$$(A + g) \circ (\text{id}_E + v) = (\text{id}_E + v) \circ A. \quad (3)$$

The map $\text{id}_E + v$ is a homeomorphism from E onto E , and both v and $w := (\text{id}_E + v)^{-1} - \text{id}_E$ are Hölder. Moreover, w is the unique bounded continuous map such that

$$A \circ (\text{id}_E + w) = (\text{id}_E + w) \circ (A + g). \quad (4)$$

If $g(0) = 0$, then also $v(0) = 0$.

A Hölder exponent α for v and w can be described explicitly (Remark 4.4 (a) and (b)). See also [1] for a recent discussion of the Hölder properties of v and w in the real case (if $g(0) = 0$).

To obtain a local linearization, following Hartman [16], we shall only require strict differentiability of f at the fixed point f . Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces over a valued field $(\mathbb{K}, |\cdot|)$, $U \subseteq E$ be open and $z \in U$. We recall from Bourbaki [4]: A map

$$f: U \rightarrow F \quad (5)$$

is called *strictly differentiable at x* if there exists a (necessarily unique) continuous linear map $f'(x): E \rightarrow F$ such that

$$\frac{\|f(y) - f(z) - f'(x)(y - z)\|_F}{\|y - z\|_E} \rightarrow 0 \quad (6)$$

if $(y, z) \in U \times U \setminus \{(u, u): u \in U\}$ tends to (x, x) . If we write $f(y) = f(x) + f'(x)(y - x) + R(y)$, then f is strictly differentiable at x with derivative $f'(x)$ if and only if R is Lipschitz on the ball $B_r^E(x)$ for small $r > 0$, and

$$\lim_{r \rightarrow 0} \text{Lip}(R|_{B_r^E(x)}) = 0 \quad (7)$$

(using standard notation as in 2.3).¹

If $E = F$ and f is strictly differentiable at $x \in U$, we call x a *hyperbolic fixed point of f* if $f(x) = x$ and $f'(x): E \rightarrow E$ is a hyperbolic automorphism.

Theorem B (Local Grobman-Hartman Theorem) *Let $(\mathbb{K}, |\cdot|)$ be an ultrametric field and E be an ultrametric Banach space over $(\mathbb{K}, |\cdot|)$. Or let $\mathbb{K} = \mathbb{R}$, $|\cdot|$ be an absolute value on \mathbb{R} which defines the usual topology on \mathbb{R} , and E be a Banach space over $(\mathbb{R}, |\cdot|)$. Let $P, Q \subseteq E$ be open and $x \in P \cap Q$. Let $f: P \rightarrow Q$ be a homeomorphism which is strictly differentiable at x , with differential $A := f'(x)$, and for which x is a hyperbolic fixed point. Then there exists an open 0-neighbourhood $U \subseteq E$ and a bi-Hölder homeomorphism $H: U \rightarrow V$ onto an open subset $V \subseteq P$, such that $H(0) = x$ and*

$$f(H(y)) = H(A(y)) \quad \text{for all } y \in U \cap A^{-1}(U). \quad (8)$$

¹Precisely this requirement on the non-linearity is also imposed in [16].

Recall that the absolute values $|\cdot|$ on \mathbb{R} defining its usual topology are precisely the p -th powers of the usual absolute value, $|\cdot| = (|\cdot|_{\mathbb{R}})^p$, with $p \in]0, 1]$. The Banach spaces E over $(\mathbb{R}, |\cdot|_{\mathbb{R}}^p)$ are also known as real p -Banach spaces in the functional-analytic literature (see [20]).

To deduce Theorem B from Theorem A, we shall cut off the nonlinearity. Since suitable cut-offs only come to mind in the real and ultrametric cases, we have to restrict attention to these situations.

In the global case, we also discuss the dependence of the conjugacies $\text{id}_E + v$ (and $\text{id}_E + w$) on f . We obtain continuous dependence of v and w when considered as elements of appropriate Hölder spaces, if \mathbb{K} is locally compact and E has finite dimension. For arbitrary \mathbb{K} and in arbitrary dimension, we get Lipschitz resp. Hölder continuous dependence of v (resp., w) with respect to the supremum norm (see Theorem 7.6 for details). For earlier results concerning parameter dependence in the real case, the reader is referred to [18, Theorem 26].

To put the requirement of strict differentiability into context, we recall: If $\mathbb{K} = \mathbb{R}$, equipped with its usual absolute value, then f as in (5) is strictly differentiable at each $x \in U$ if and only if f is continuously Fréchet differentiable ([4, 2.3.3], [6, Theorem 3.8.1]). If $(\mathbb{K}, |\cdot|)$ is arbitrary and f is C^2 in the sense of [3], then f is strictly differentiable at each x [7, Proposition 3.4]. If $(\mathbb{K}, |\cdot|)$ is a complete ultrametric field and E of finite dimension, then f is strictly differentiable at each x if and only if f is C^1 in the sense of [3] (see [10, Appendix C]), hence if and only if it is C^1 in the usual sense of non-archimedean analysis in finite-dimensional spaces (as in [28], [29]); see [8].

In the classical real case, it is known that conjugacies cannot be chosen locally Lipschitz in general (see [2], cited from [31]). In particular, they need not be C^1 (although the C^1 -property – and higher differentiability properties – can be guaranteed under suitable non-resonance conditions [30]). The investigation of the possible continuity and differentiability properties of local conjugacies (e.g., differentiability at the fixed point) remains an active area of research (see [1], [14], [25], [26], [31] for some recent work). The current article provides a foundation for the study of such refined questions also in the non-archimedean case.

Linearization problems also arise in Lie theory, in connection with simple Lie

contraction groups: Consider a non-trivial analytic Lie group G over a local field \mathbb{K} of positive characteristic, admitting an automorphism $f: G \rightarrow G$ which is contractive (i.e., $f^n(x) \rightarrow 1$ as $n \rightarrow \infty$, for each $x \in G$), and such that G does not have f -stable Lie subgroups except for $\{1\}$ and G . Then G is abelian. It is an open question [9] whether (G, f) is isomorphic to some \mathbb{K}^n with a linear automorphism (as in characteristic 0). The question is equivalent to the existence of an analytic local linearization which is, moreover, a group homomorphism to \mathbb{K}^n .

Of course, the above concept of hyperbolicity is also useful in other regards. For example, as in the real case, a stable manifold can be constructed around each hyperbolic fixed point (if f is analytic and the adapted norm is ultrametric [11]). Further ideas and facts from hyperbolic dynamics are waiting to be transferred into non-archimedean analysis.

2 Preliminaries and notation

In this section, we fix some notation and compile facts and preparatory results for later use.

2.1 Given a metric space (X, d) , $r > 0$ and $x \in X$, we define $B_r^X(x) := \{y \in X: d(x, y) < r\}$ and $\overline{B}_r^X(x) := \{y \in X: d(x, y) \leq r\}$. As usual, a normed space $(E, \|\cdot\|)$ over a valued field $(\mathbb{K}, |\cdot|)$ is called a Banach space if it is complete. If, moreover, $(\mathbb{K}, |\cdot|)$ is an ultrametric field and also $\|\cdot\|$ satisfies the ultrametric inequality $\|x + y\| \leq \max\{\|x\|, \|y\|\}$, then $(E, \|\cdot\|)$ is called an *ultrametric* Banach space (see [27] for further information). The ultrametric inequality implies that

$$\|x + y\| = \|y\| \quad \text{for all } x, y \in E \text{ such that } \|x\| < \|y\|. \quad (9)$$

If $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are normed spaces over a valued field $(\mathbb{K}, |\cdot|)$ and $A: E \rightarrow F$ a continuous linear map, then its operator norm is defined as $\|A\| := \sup\{\|Ax\|_F / \|x\|_E: 0 \neq x \in E\} \in [0, \infty[$.

2.2 If $f: X \rightarrow E$ is a bounded map to a normed space $(E, \|\cdot\|)$ over a valued field $(\mathbb{K}, |\cdot|)$, we write $\|f\|_\infty := \sup\{\|f(x)\|: x \in X\}$ for its supremum norm. Given a topological space X , we write $BC(X, E)$ for the set of bounded, continuous functions from X to E . This is a normed space with respect to the supremum norm, and a Banach space (respectively, an ultrametric Banach space) if so is E .

2.3 As usual, we call a map $f: X \rightarrow Y$ between metric spaces (X, d_X) and (Y, d_Y) (globally) *Hölder* of exponent $\alpha \in]0, \infty[$ if there exists $L \in [0, \infty[$ such that $d_Y(f(x), f(y)) \leq L d_X(x, y)^\alpha$ for all $x, y \in X$. We let $\text{Lip}_\alpha(f)$ be the minimum choice of L . If f is bijective and both f and f^{-1} are Hölder (of exponent α), we call f a bi-Hölder homeomorphism (of exponent α). Hölder maps of exponent 1 are called Lipschitz and we abbreviate $\text{Lip}(f) := \text{Lip}_1(f)$. Thus $\text{Lip}(A) = \|A\|$ for continuous linear maps. We write $L_\alpha(X, Y)$ for the set of all Hölder maps $f: X \rightarrow Y$ of exponent α . If $(E, \|\cdot\|_E)$ is a Banach space over a valued field $(\mathbb{K}, |\cdot|)$, then also

$$BL_\alpha(X, E) := L_\alpha(X, E) \cap BC(X, E)$$

is a Banach space, with respect to the norm $\|f\|_\alpha := \max\{\|f\|_\infty, \text{Lip}_\alpha(f)\}$. If, moreover, X is compact, then $BL_\alpha(X, E) = L_\alpha(X, E)$ and thus $\|\cdot\|_\alpha$ makes $L_\alpha(X, E)$ a Banach space.

Lemma 2.4 *Let (X, d_X) , (Y, d_Y) and (Z, d_Z) be metric spaces, $f: X \rightarrow Y$ be Hölder of exponent α , and $g: Y \rightarrow Z$ be Hölder of exponent β . Then $g \circ f: X \rightarrow Z$ is Hölder of exponent $\alpha\beta$, and*

$$\text{Lip}_{\alpha\beta}(g \circ f) \leq \text{Lip}_\beta(g) (\text{Lip}_\alpha(f))^\beta.$$

Proof. For $x, y \in X$, we have $d_Z(g(f(x)), g(f(y))) \leq \text{Lip}_\beta(g) d_Y(f(x), f(y))^\beta \leq \text{Lip}_\beta(g) \text{Lip}_\alpha(f)^\beta d_X(x, y)^{\alpha\beta}$. \square

Lemma 2.5 *Let (X, d_X) , (Y, d_Y) be metric spaces, $\alpha \geq \beta > 0$ and $f: X \rightarrow Y$ be a Hölder map of exponent α , which is bounded in the sense that*

$$\text{spread}(f) := \sup\{d_Y(f(x), f(y)) : x, y \in X\} < \infty.$$

Then f is also Hölder of exponent β , and

$$\text{Lip}_\beta(f) \leq \max\{\text{Lip}_\alpha(f), \text{spread}(f)\}. \quad (10)$$

Proof. Let $x, y \in X$. If $d_X(x, y) \leq 1$, then

$$d_Y(f(x), f(y)) \leq \text{Lip}_\alpha(f) d_X(x, y)^\alpha \leq \text{Lip}_\alpha(f) d_X(x, y)^\beta. \quad (11)$$

If $d_X(x, y) \geq 1$, then

$$d_Y(f(x), f(y)) \leq \text{spread}(f) \leq \text{spread}(f) d_X(x, y)^\beta. \quad (12)$$

Now (10) follows from (11) and (12). \square

Lemma 2.6 *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed spaces over a valued field $(\mathbb{K}, |\cdot|)$, $h: E \rightarrow F$ be a bounded Lipschitz map and $v: E \rightarrow E$ be a map which is Hölder of some exponent $\alpha \in]0, 1]$. Then also the map $h \circ (\text{id}_E + v): E \rightarrow F$ is Hölder of exponent α , and*

$$\text{Lip}_\alpha(h \circ (\text{id}_E + v)) \leq \max\{\text{Lip}(h)(1 + \text{Lip}_\alpha(v)), \text{spread}(h)\}.$$

In particular, $\text{Lip}_\alpha(h \circ (\text{id}_E + v)) \leq \max\{\text{Lip}(h)(1 + \text{Lip}_\alpha(v)), 2\|h\|_\infty\}$.

Proof. Let $x, y \in E$. If $\|y - x\|_E \leq 1$, then $\|y - x\|_E \leq \|y - x\|_E^\alpha$ and hence

$$\begin{aligned} \|h(y + v(y)) - h(x + v(x))\|_F &\leq \text{Lip}(h)\|y + v(y) - x - v(x)\|_E \\ &\leq \text{Lip}(h)(\|y - x\|_E + \text{Lip}_\alpha(v)\|y - x\|_E^\alpha) \\ &\leq \text{Lip}(h)(1 + \text{Lip}_\alpha(v))\|y - x\|_E^\alpha. \end{aligned}$$

If $\|y - x\|_E \geq 1$, we have $\|h(y + v(y)) - h(x + v(x))\|_F \leq \text{spread}(h) \leq \text{spread}(h)\|y - x\|_E^\alpha$. The assertion follows from the preceding estimates. \square

Lemma 2.7 *Let $(E, \|\cdot\|)$ be a normed space over a valued field $(\mathbb{K}, |\cdot|)$, (X, d) be a metric space and $\xi: X \rightarrow \mathbb{K}$ and $f: X \rightarrow E$ be bounded, Lipschitz maps. Then also the pointwise product ξf is bounded and Lipschitz, with*

$$\text{Lip}(\xi f) \leq \text{Lip}(\xi)\|f\|_\infty + \|\xi\|_\infty \text{Lip}(f).$$

Proof. $\|\xi(y)f(y) - \xi(x)f(x)\| \leq |\xi(y) - \xi(x)|\|f(y)\| + |\xi(x)|\|f(y) - f(x)\|$. \square

Lemma 2.8 *Let $(E, \|\cdot\|)$ be a Banach space over a valued field $(\mathbb{K}, |\cdot|)$ (such that $E \neq \{0\}$) and $A: E \rightarrow E$ be an automorphism of topological vector spaces. Moreover, let $v: E \rightarrow E$ be a Lipschitz map such that $\text{Lip}(v) < \frac{1}{\|A^{-1}\|}$. Then the map $f := A + v: E \rightarrow E$ is a homeomorphism, and $f^{-1}: E \rightarrow E$ is Lipschitz with*

$$\text{Lip}(f^{-1}) \leq \frac{1}{\|A^{-1}\|^{-1} - \text{Lip}(v)} \quad \text{and} \quad (13)$$

$$\text{Lip}(f^{-1} - A^{-1}) \leq \frac{\|A^{-1}\|}{\|A^{-1}\|^{-1} - \text{Lip}(v)} \text{Lip}(v). \quad (14)$$

If v is bounded, then also $w := f^{-1} - A^{-1}$ is bounded, and $\|w\|_\infty \leq \|A^{-1}\| \|v\|_\infty$.

Proof. Set $a := \|A^{-1}\|^{-1} - \text{Lip}(v) > 0$. By the Lipschitz Inverse Function Theorem (see [10, Theorem 5.3]), the restriction $f_r := f|_{B_r^E(0)}$ is injective for each $r > 0$, whence f is injective. By the same theorem, the inverse map $(f_r)^{-1}: f(B_r^E(0)) \rightarrow E$ is Lipschitz with $\text{Lip}(f_r^{-1}) \leq a^{-1}$. Hence also $f^{-1}: f(E) \rightarrow E$ is Lipschitz, with $\text{Lip}(f^{-1}) \leq a^{-1}$, and thus (13) holds. In particular, f is a homeomorphism onto its image. By the cited theorem, $f(B_r^E(0)) \supseteq B_{ar}^E(f(0))$ for each r . Hence $f(E) \supseteq \bigcup_{r>0} B_{ar}^E(f(0)) = E$, whence f is surjective. To complete the proof, write $w := f^{-1} - A^{-1}$. Then $\text{id}_E = (A+v) \circ (A+v)^{-1} = (A+v) \circ (A^{-1}+w) = \text{id}_E + A \circ w + v \circ (A^{-1}+w) = \text{id}_E + A \circ w + v \circ f^{-1}$ and thus

$$w = -A^{-1} \circ v \circ f^{-1}. \quad (15)$$

Hence $\text{Lip}(w) \leq \text{Lip}(A^{-1}) \text{Lip}(v) \text{Lip}(f^{-1}) = \|A^{-1}\| \text{Lip}(v) \text{Lip}(f^{-1})$. If we combine this estimate with (13), we obtain (14). Finally, assuming that v is bounded, (15) shows that also w is bounded, with $\|w\|_\infty \leq \|A^{-1}\| \|v\|_\infty$. \square

3 Passage from one perturbation to another

In this section, we construct conjugacies from one perturbation of a given hyperbolic automorphism to another.

Lemma 3.1 *Let $E \neq \{0\}$ be a Banach space over a valued field $(\mathbb{K}, |\cdot|)$, $A: E \rightarrow E$ be a hyperbolic automorphism, and $\|\cdot\|$ be an adapted norm on E . Let $g = (g_s, g_u)$, $h = (h_s, h_u): E \rightarrow E = E_s \oplus E_u$ be bounded Lipschitz maps such that*

$$\text{Lip}(h) < \|A^{-1}\|^{-1} \quad \text{and} \quad (16)$$

$$\Lambda := \max \{ \|A_2^{-1}\| (1 + \text{Lip}(g_u)), \|A_1\| + \text{Lip}(g_s) \} < 1, \quad (17)$$

with $A_1 := A|_{E_s}: E_s \rightarrow E_s$ and $A_2 := A|_{E_u}$. Then there exists a unique bounded continuous map $v: E \rightarrow E$ such that

$$(\text{id}_E + v) \circ (A + h) = (A + g) \circ (\text{id}_E + v). \quad (18)$$

It satisfies

$$\|v\|_\infty \leq \frac{\max \{ \|h_s\|_\infty + \|g_s\|_\infty, \|A_2^{-1}\| (\|h_u\|_\infty + \|g_u\|_\infty) \}}{1 - \Lambda}. \quad (19)$$

If $g(0) = h(0) = 0$, then also $v(0) = 0$.

Proof. As a consequence of (16), $A+h: E \rightarrow E$ is a homeomorphism, whose inverse $(A+h)^{-1}$ is Lipschitz with $\text{Lip}((A+h)^{-1}) \leq (\|A^{-1}\|^{-1} - \text{Lip}(h))^{-1}$ (see Lemma 2.8). For a bounded continuous function $v: E \rightarrow E$, (18) is equivalent to $A^{-1} \circ (\text{id}_E + v) \circ (A+h) = A^{-1} \circ (A+g) \circ (\text{id}_E + v)$, which in turn is equivalent to

$$v = A^{-1} \circ h + A^{-1} \circ v \circ (A+h) - A^{-1} \circ g \circ (\text{id}_E + v). \quad (20)$$

Let $\pi_s: E \rightarrow E_s$ and $\pi_u: E \rightarrow E_u$ be the projections onto the stable and unstable subspace of E , respectively. In the following, we identify a function $k: E \rightarrow E$ with the pair (k_s, k_u) of its components $k_s := \pi_s \circ k$ and $k_u := \pi_u \circ k$. Then $BC(E, E) = BC(E, E_s) \oplus BC(E, E_u)$ as a Banach space (if we take the maximum norm on the right hand side). If $v = (v_s, v_u)$, then (20) holds if and only if both (21) and (22) are satisfied:

$$v_s = A_1^{-1} \circ h_s + A_1^{-1} \circ v_s \circ (A+h) - A_1^{-1} \circ g_s \circ (\text{id}_E + v) \quad (21)$$

$$v_u = A_2^{-1} \circ h_u + A_2^{-1} \circ v_u \circ (A+h) - A_2^{-1} \circ g_u \circ (\text{id}_E + v) =: \theta_2(v). \quad (22)$$

Moreover, (21) is satisfied if and only if

$$v_s = A_1 \circ v_s \circ (A+h)^{-1} - h_s \circ (A+h)^{-1} + g_s \circ (\text{id}_E + v) \circ (A+h)^{-1} =: \theta_1(v). \quad (23)$$

Thus (18) holds if and only if $v \in BC(E, E)$ is a fixed point of the self-map

$$\theta := (\theta_1, \theta_2): BC(E, E) \rightarrow BC(E, E_s) \oplus BC(E, E_u) = BC(E, E)$$

of the Banach space $BC(E, E)$. *We claim that θ is a contraction, with $\text{Lip}(\theta) \leq \Lambda$.* If this is true, then θ will have a unique fixed point (the unique v we seek), by Banach's Fixed Point Theorem [21, Theorem 3.4.1]. Starting the iterative approximation of v with the zero-function $v_0 := 0: E \rightarrow E$, the standard *a priori* estimate (see [21, Proposition 3.4.4]) gives

$$\|v\|_\infty = \|v - v_0\|_\infty \leq \frac{\|\theta(v_0) - v_0\|_\infty}{1 - \Lambda} = \frac{\|\theta(v_0)\|_\infty}{1 - \Lambda}$$

and applying now the triangle inequality to the individual summands in $\|\theta(v_0)\|_\infty = \|\theta(0)\|_\infty = \max\{\|\theta_1(0)\|_\infty, \|\theta_2(0)\|_\infty\}$ (as in (23) and (22)), we obtain (19). If $g(0) = h(0) = 0$, we have $\theta^n(v_0)(0) = 0$ for each $n \in \mathbb{N}_0$, by a trivial induction. Hence also $v = \lim_{n \rightarrow \infty} \theta^n(v_0)$ vanishes at 0.

To establish the claim, we need only show that both $\text{Lip}(\theta_1), \text{Lip}(\theta_2) \leq \Lambda$, because $\text{Lip}(\theta) = \max\{\text{Lip}(\theta_1), \text{Lip}(\theta_2)\}$. Given $v, w \in BC(E, E)$, we have

$$\begin{aligned} \|\theta_2(v) - \theta_2(w)\|_\infty &\leq \|A_2^{-1} \circ (v_u - w_u) \circ (A + h)\|_\infty \\ &\quad + \|A_2^{-1} \circ g_u \circ (\text{id}_E + v) - A_2^{-1} \circ g_u \circ (\text{id}_E + w)\|_\infty. \end{aligned}$$

Since $\|A_2^{-1} \circ (v_u - w_u) \circ (A + h)\|_\infty \leq \|A_2^{-1}\| \cdot \|v - w\|_\infty$ and

$$\|A_2^{-1} \circ g_u \circ (\text{id}_E + v) - A_2^{-1} \circ g_u \circ (\text{id}_E + w)\|_\infty \leq \|A_2^{-1}\| \cdot \text{Lip}(g_u) \cdot \|v - w\|_\infty,$$

we get $\|\theta_2(v) - \theta_2(w)\|_\infty \leq \|A_2^{-1}\|(1 + \text{Lip}(g_u))\|v - w\|_\infty$ and thus

$$\text{Lip}(\theta_2) \leq \|A_2^{-1}\|(1 + \text{Lip}(g_u)) \leq \Lambda \quad (24)$$

(using (17)). Moreover,

$$\begin{aligned} \|\theta_1(v) - \theta_1(w)\|_\infty &\leq \\ &\|A_1 \circ (v_s - w_s)\|_\infty + \|g_s \circ (\text{id}_E + v) \circ (A + h)^{-1} - g_s \circ (\text{id}_E + w) \circ (A + h)^{-1}\|_\infty. \end{aligned}$$

As $\|g_s \circ (\text{id}_E + v) \circ (A + h)^{-1} - g_s \circ (\text{id}_E + w) \circ (A + h)^{-1}\|_\infty \leq \text{Lip}(g_s)\|v - w\|_\infty$ and $\|A_1 \circ (v_s - w_s)\|_\infty \leq \|A_1\| \cdot \|v - w\|_\infty$, we obtain

$$\text{Lip}(\theta_1) \leq \|A_1\| + \text{Lip}(g_s) \leq \Lambda \quad (25)$$

(using (17) again). Thus

$$\text{Lip}(\theta) \leq \Lambda, \quad (26)$$

which completes the proof. \square

Lemma 3.2 *In the situation of Lemma 3.1, assume that also*

$$\text{Lip}(g) < \|A^{-1}\|^{-1}, \quad (27)$$

$$\|A_2^{-1}\|(1 + \text{Lip}(h_u)) < 1, \quad \text{and} \quad \|A_1\| + \text{Lip}(h_s) < 1 \quad (28)$$

hold. Then the map $\text{id}_E + v: E \rightarrow E$ is a homeomorphism. Moreover, $w := (\text{id}_E + v)^{-1} - \text{id}_E: E \rightarrow E$ is the unique bounded continuous map such that

$$(\text{id}_E + w) \circ (A + g) = (A + h) \circ (\text{id}_E + w). \quad (29)$$

Proof. In view of (27) and (28), we can apply Lemma 3.1 with reversed roles of g and h , and obtain a unique bounded continuous map $w: E \rightarrow E$ such that (29) holds. Then

$$(\text{id}_E + v) \circ (\text{id}_E + w) = \text{id}_E + f,$$

where $f := w + v \circ (\text{id}_E + w)$ is continuous and bounded. Now

$$\begin{aligned} (\text{id}_E + f) \circ (A + g) &= (\text{id}_E + v) \circ (\text{id}_E + w) \circ (A + g) \\ &= (\text{id}_E + v) \circ (A + h) \circ (\text{id}_E + w) \\ &= (A + g) \circ (\text{id}_E + v) \circ (\text{id}_E + w) = (A + g) \circ (\text{id}_E + f), \end{aligned}$$

using (29) to obtain the second equality and (18) for the third. Since also $(\text{id}_E + 0) \circ (A + g) = (A + g) \circ (\text{id}_E + 0)$, the uniqueness property in Lemma 3.1 (applied to g and g in place of g and h) shows that $f = 0$ and therefore $(\text{id}_E + v) \circ (\text{id}_E + w) = \text{id}_E$. Reversing the roles of g and h , the same argument gives $(\text{id}_E + w) \circ (\text{id}_E + v) = \text{id}_E$. Thus $\text{id}_E + v$ is invertible with $(\text{id}_E + v)^{-1} = \text{id}_E + w$. The assertions follow. \square

4 Hölder property of the conjugacies

We now show that the mappings v constructed in Section 3 are Hölder.

Lemma 4.1 *Let (X, d_X) and (Y, d_Y) be metric spaces, $\alpha > 0$ and $(f_j)_{j \in J}$ be a net in $L_\alpha(X, Y)$ which converges pointwise to a function $f: X \rightarrow Y$. If*

$$\lambda := \sup\{\text{Lip}_\alpha(f_j) : j \in J\} < \infty,$$

then $f \in L_\alpha(X, Y)$ and $\text{Lip}_\alpha(f) \leq \lambda$.

Proof. Given $x, y \in X$, we have $d_Y(f_j(x), f_j(y)) \leq \lambda d_X(x, y)^\alpha$ for all $j \in J$. Passing to the limit, we obtain $d_Y(f(x), f(y)) \leq \lambda d_X(x, y)^\alpha$. \square

Lemma 4.2 *In the situation of Lemma 3.1, let $k := (A + h)^{-1}$ and assume that*

$$\text{Lip}_\alpha(h_s \circ k) + \text{Lip}(k)^\alpha(\varepsilon \|A_1\| + \max\{\text{Lip}(g_s)(1 + \varepsilon), \text{spread}(g_s)\}) \leq \varepsilon \quad (30)$$

and

$$\begin{aligned} \text{Lip}_\alpha(A_2^{-1} \circ h_u) + \varepsilon \|A_2^{-1}\| \text{Lip}(A + h)^\alpha \\ + \|A_2^{-1}\| \max\{\text{Lip}(g_u)(1 + \varepsilon), \text{spread}(g_u)\} \leq \varepsilon \end{aligned} \quad (31)$$

for a given number $\alpha \in]0, 1[$. Then the bounded continuous map $v: E \rightarrow E$ determined by (18) is Hölder of exponent α , and

$$\text{Lip}_\alpha(v) \leq \varepsilon.$$

Proof. We retain the notation introduced in the proof of Lemma 3.1; in particular, we shall use the contraction $\theta = (\theta_1, \theta_2): BC(E, E) \rightarrow BC(E, E)$ introduced there. By Lemma 4.1, the (non-empty) set

$$Y := \{f \in BC(E, E) \cap L_\alpha(E, E): \text{Lip}_\alpha(f) \leq \varepsilon\} \quad (32)$$

is closed in $BC(E, E)$, and hence a complete metric space with the metric induced by that on $BC(E, E)$, $d_\infty(u, w) := \|u - w\|_\infty$. We claim that $\theta(Y) \subseteq Y$. If this is true, then the Banach Fixed Point Theorem provides a unique fixed point $y \in Y$ for the contraction $\theta|_Y: Y \rightarrow Y$ of (Y, d_∞) . Then y has to coincide with the unique fixed point $v \in BC(E, E)$ of θ (the map v determined by (18)), and thus $v = y \in Y$, whence all assertions of the lemma hold. Since

$$L_\alpha(E, E) = L_\alpha(E, E_s) \oplus L_\alpha(E, E_u) \quad \text{and} \quad (33)$$

$$\text{Lip}_\alpha(f) = \max\{\text{Lip}_\alpha(f_s), \text{Lip}_\alpha(f_u)\} \quad (34)$$

for $f = (f_s, f_u) \in L_\alpha(E, E)$, to establish the claim we need only show that both $\theta_1(v)$ and $\theta_2(v)$ are Hölder of exponent α for each $v \in Y$, and $\text{Lip}_\alpha(\theta_1(v)), \text{Lip}_\alpha(\theta_2(v)) \leq \varepsilon$. In view of Lemmas 2.4, 2.5 and 2.6, all three summands in (23) are Hölder of exponent α . Now

$$\text{Lip}_\alpha(\theta_1(v)) \leq \text{Lip}_\alpha(A_1 \circ v_s \circ k) + \text{Lip}_\alpha(h_s \circ k) + \text{Lip}_\alpha(g_s \circ (\text{id}_E + v) \circ k), \quad (35)$$

where $\text{Lip}_\alpha(A_1 \circ v_s \circ k) \leq \|A_1\| \text{Lip}_\alpha(v_s) \text{Lip}(k)^\alpha \leq \varepsilon \|A_1\| \text{Lip}(k)^\alpha$ by Lemma 2.4 and

$$\begin{aligned} \text{Lip}_\alpha(g_s \circ (\text{id}_E + v) \circ k) &\leq \text{Lip}_\alpha(g_s \circ (\text{id}_E + v)) \text{Lip}(k)^\alpha \\ &\leq \max\{\text{Lip}(g_s)(1 + \text{Lip}_\alpha(v)), \text{spread}(g_s)\} \text{Lip}(k)^\alpha \\ &\leq \max\{\text{Lip}(g_s)(1 + \varepsilon), \text{spread}(g_s)\} \text{Lip}(k)^\alpha \end{aligned}$$

by Lemmas 2.4 and 2.6. To obtain an upper bound for $\text{Lip}_\alpha(\theta_1(v))$, we substitute the preceding estimates into (35). The upper bound so obtained is the left hand side of (30) and hence $\leq \varepsilon$ by hypotheses. Thus $\text{Lip}_\alpha(\theta_1(v)) \leq \varepsilon$. Similarly, Lemmas 2.4, 2.5 and 2.6 show that all three summands in (22) are Hölder of exponent α . Now

$$\begin{aligned} \text{Lip}_\alpha(\theta_2(v)) &\leq \text{Lip}_\alpha(A_2^{-1} \circ h_u) + \text{Lip}_\alpha(A_2^{-1} \circ v_u \circ (A + h)) \\ &\quad + \text{Lip}_\alpha(A_2^{-1} \circ g_u \circ (\text{id}_E + v)); \end{aligned} \quad (36)$$

here $\text{Lip}_\alpha(A_2^{-1} \circ v_u \circ (A + h)) \leq \|A_2^{-1}\| \text{Lip}_\alpha(v_u) \text{Lip}(A + h)^\alpha \leq \varepsilon \|A_2^{-1}\| \text{Lip}(A + h)^\alpha$ by Lemma 2.4 and

$$\begin{aligned} \text{Lip}_\alpha(A_2^{-1} \circ g_u \circ (\text{id}_E + v)) &\leq \|A_2^{-1}\| \max\{\text{Lip}(g_u)(1 + \text{Lip}_\alpha(v)), \text{spread}(g_u)\} \\ &\leq \|A_2^{-1}\| \max\{\text{Lip}(g_u)(1 + \varepsilon), \text{spread}(g_u)\} \end{aligned}$$

by Lemmas 2.4 and 2.6. Combining (36) with the preceding estimates, we get the left hand side of (31) as an upper bound for $\text{Lip}_\alpha(\theta_2(v))$. Hence also $\text{Lip}_\alpha(\theta_2(v)) \leq \varepsilon$ and thus $\theta(v) \in Y$, which completes the proof. \square

The conditions (30) and (31) describe exactly what we need in the proof, but they are somewhat elusive. They can be replaced by stronger (but more tangible) hypotheses, which we now state.

Lemma 4.3 *If g and h are as in Lemma 3.1 and*

$$\begin{aligned} \frac{\varepsilon \|A_1\|}{(\|A^{-1}\|^{-1} - \text{Lip}(h))^\alpha} + \max\left\{\frac{\text{Lip}(h_s)}{\|A^{-1}\|^{-1} - \text{Lip}(h)}, \text{spread}(h_s)\right\} \\ + \frac{\max\{\text{Lip}(g_s)(1 + \varepsilon), \text{spread}(g_s)\}}{(\|A^{-1}\|^{-1} - \text{Lip}(h))^\alpha} \leq \varepsilon \end{aligned} \quad (37)$$

as well as

$$\begin{aligned} \|A_2^{-1}\| \max\{\text{Lip}(h_u), \text{spread}(h_u)\} + \varepsilon \|A_2^{-1}\| (\|A\| + \text{Lip}(h))^\alpha \\ + \|A_2^{-1}\| \max\{\text{Lip}(g_u)(1 + \varepsilon), \text{spread}(g_u)\} \leq \varepsilon, \end{aligned} \quad (38)$$

then the conditions (30) and (31) from Lemma 4.2 are satisfied. In particular, if $\alpha \in]0, 1[$ and $\varepsilon > 0$ are given and we choose $\delta > 0$ so small that

$$\delta < \|A^{-1}\|^{-1} \quad \|A_2^{-1}\|(1 + \delta) < 1, \quad \|A_1\| + \delta < 1, \quad (39)$$

$$2\|A_2^{-1}\|\delta + \varepsilon\|A_2^{-1}\|(\|A\| + \delta)^\alpha + \|A_2^{-1}\| \max\{\delta(1 + \varepsilon), 2\delta\} \leq \varepsilon, \quad \text{and} \quad (40)$$

$$\frac{\varepsilon\|A_1\|}{(\|A^{-1}\|^{-1} - \delta)^\alpha} + \max\left\{\frac{\delta}{\|A^{-1}\|^{-1} - \delta}, 2\delta\right\} + \frac{\max\{\delta(1 + \varepsilon), 2\delta\}}{(\|A^{-1}\|^{-1} - \delta)^\alpha} \leq \varepsilon, \quad (41)$$

then conditions (16), (17), (30) and (31) are satisfied for all bounded, Lipschitz maps $g, h: E \rightarrow E$ with

$$\max\{\|g\|_\infty, \text{Lip}(g)\} \leq \delta \quad \text{and} \quad \max\{\|h\|_\infty, \text{Lip}(h)\} \leq \delta. \quad (42)$$

Proof. Let $k := (A + h)^{-1}$, as in Lemma 4.2. Then

$$\text{Lip}(k) \leq \frac{1}{\|A^{-1}\|^{-1} - \text{Lip}(h)}, \quad (43)$$

by (13). Next,

$$\begin{aligned} \text{Lip}_\alpha(h_s \circ k) &\leq \max\{\text{Lip}(h_s \circ k), \text{spread}(h_s \circ k)\} \\ &\leq \max\left\{\frac{\text{Lip}(h_s)}{\|A^{-1}\|^{-1} - \text{Lip}(h)}, \text{spread}(h_s)\right\}, \end{aligned} \quad (44)$$

using Lemma 2.5, Lemma 2.4, and the estimate (43). We also have

$$\text{Lip}_\alpha(A_2^{-1} \circ h_u) \leq \|A_2^{-1}\| \text{Lip}_\alpha(h_u) \leq \|A_2^{-1}\| \max\{\text{Lip}(h_u), \text{spread}(h_u)\}, \quad (45)$$

using Lemmas 2.4 and 2.5. Finally, we have

$$\varepsilon\|A_2^{-1}\| \text{Lip}(A + h)^\alpha \leq \varepsilon\|A_2^{-1}\|(\|A\| + \text{Lip}(h))^\alpha. \quad (46)$$

In view of (43)–(46), it is clear that (37) implies (30) and (38) implies (31). The final assertion of the lemma is now obvious, using that $\text{spread}(f) \leq 2\|f\|_\infty$ for all bounded maps f between normed spaces. \square

Remark 4.4 (a) Note that, given h, g as in Lemma 3.1, one can always find $\alpha \in]0, 1[$ and $\varepsilon > 0$ such that (37) and (38) (and hence also (30) and (31)) are satisfied. In fact, we have $1 - \|A_1\| - \text{Lip}(g_s) > 0$ by (17) and hence also

$$\Delta_{g,h} := 1 - \frac{\|A_1\| + \text{Lip}(g_s)}{(\|A^{-1}\|^{-1} - \text{Lip}(h))^\alpha} > 0 \quad (47)$$

for sufficiently small $\alpha \in]0, 1[$. Instead of (37), to simplify the calculation let us impose a stronger condition by replacing the second maximum $\max\{\text{Lip}(g_s)(1 + \varepsilon), \text{spread}(g_s)\}$ in (37) by the larger term

$$\max\{\text{Lip}(g_s), \text{spread}(g_s)\} + \varepsilon \text{Lip}(g_s) .$$

We can then solve for ε and see that the strengthened inequality is equivalent to

$$\varepsilon \geq \frac{\max\left\{\frac{\text{Lip}(h_s)}{\|A^{-1}\|^{-1} - \text{Lip}(h)}, \text{spread}(h_s)\right\} + \frac{\max\{\text{Lip}(g_s), \text{spread}(g_s)\}}{(\|A^{-1}\|^{-1} - \text{Lip}(h))^\alpha}}{\Delta_{g,h}}. \quad (48)$$

Also, we have $1 - \|A_2^{-1}\|(1 + \text{Lip}(g_u)) > 0$ by (17) and hence

$$\delta_{g,h} := 1 - \|A_2^{-1}\|((\|A\| + \text{Lip}(h))^\alpha + \text{Lip}(g_u)) > 0 \quad (49)$$

for sufficiently small $\alpha \in]0, 1[$. Likewise, replacing $\|A_2^{-1}\|$ times the second maximum in (38) by

$$\|A_2^{-1}\| \max\{\text{Lip}(g_u), \text{spread}(g_u)\} + \varepsilon \|A_2^{-1}\| \text{Lip}(g_u) ,$$

we obtain a stronger condition equivalent to

$$\varepsilon \geq \frac{\|A_2^{-1}\|(\max\{\text{Lip}(h_u), \text{spread}(h_u)\} + \max\{\text{Lip}(g_u), \text{spread}(g_u)\})}{\delta_{g,h}}. \quad (50)$$

Now choose ε so large that both (48) and (50) hold.

- (b) Given g and h as in Lemma 3.1, we can actually find $\alpha \in]0, 1[$ and $\varepsilon > 0$ such that (37) and (38) are satisfied simultaneously for (g, h) and (h, g) (i.e., with reversed roles of h and g): Simply proceed as in (a) for both pairs, and replace the values of α obtained by their minimum. Then choose an ε for this α in both cases, and replace the two values of ε by their maximum.
- (c) Note that we did not need to assume that $g(0) = 0$ or $h(0) = 0$ in our previous results (although, of course, this case is of primary interest).
- (d) Because $\text{spread}(f) \leq 2\|f\|_\infty$, one can replace $\text{spread}(f)$ with $2\|f\|_\infty$ in (37) and (38) for $f = g_s, g_u, h_s, h_u$, and obtains simpler-looking, alternative conditions which also imply (30) and (31).

5 Proof of Theorem A

The assertions of the theorem are covered by Lemmas 3.1, 3.2 and 4.2 and Remark 4.4 (a), setting $h := 0$ there.

6 Proof of Theorem B

We give the proof in a form which can be re-used in a later work (or version) in the study of parameter dependence. Avoiding only a trivial case, assume $E \neq \{0\}$. After a translation, we may (and will) assume that $x = 0$. After shrinking P , we may also assume that $P = B_r^E(0)$ for some $r > 0$. Write $f(y) = f(0) + f'(0)(y) + R(y)$; thus

$$f(y) = A(y) + R(y) \quad \text{for all } y \in B_r^E(0),$$

with $A := f'(0)$. Let $E = E_s \oplus E_u$ with respect to A and $\|\cdot\|$ be an adapted norm on E .

6.1 If \mathbb{K} and E are ultrametric, then also the adapted norm $\|\cdot\|$ on E can (and will) be chosen ultrametric (see Appendix A). In this case, we define $R_s: E \rightarrow E$ for $s \in]0, r]$ via

$$R_s(y) := \begin{cases} R(y) & \text{if } y \in B_s^E(0); \\ 0 & \text{else.} \end{cases}$$

Choose s so small that $R|_{B_s^E(0)}$ is Lipschitz (see (7)). If $y, z \in B_s^E(0)$, then $\|R_s(z) - R_s(y)\| = \|R(z) - R(y)\| \leq \text{Lip}(R|_{B_s^E(0)})\|z - y\|$. If $y, z \in E \setminus B_s^E(0)$, then $\|R_s(z) - R_s(y)\| = 0$. If $z \in B_s^E(0)$ and $y \in E \setminus B_s^E(0)$, then $\|z - y\| = \|y\| > \|z\|$ by (9) and thus $\|R_s(z) - R_s(y)\| = \|R(z)\| = \|R(z) - R(0)\| \leq \text{Lip}(R|_{B_s^E(0)})\|z\| \leq \text{Lip}(R|_{B_s^E(0)})\|z - y\|$. Hence R_s is Lipschitz, with

$$\text{Lip}(R_s) \leq \text{Lip}(R|_{B_s^E(0)}) \quad (51)$$

(and in fact equality holds).

6.2 In the real case, let $\eta: [0, \infty[\rightarrow [0, 1]$ be a Lipschitz function (with respect to the ordinary absolute value on \mathbb{R}) such that $\eta|_{[0,1]} = 1$ and $\eta(t) = 0$ for $t \geq 2$. Then

$$\text{Lip}(\eta) \geq 1. \quad (52)$$

For $s \in]0, r/3]$, define

$$\xi_s: E \rightarrow [0, 1], \quad \xi_s(y) := \eta(\|y\|/s)$$

and

$$R_s(y) := \begin{cases} \xi_s(y)R(y) & \text{if } y \in B_{3s}^E(0); \\ 0 & \text{else.} \end{cases}$$

Choose s so small that $R|_{B_{3s}^E(0)}$ is Lipschitz. Then

$$\text{Lip}(R_s) \leq (1 + 3 \text{Lip}(\eta)) \text{Lip}(R|_{B_{3s}^E(0)}), \quad (53)$$

by the following arguments. First,

$$\begin{aligned} \text{Lip}(R_s|_{B_{3s}^E(0)}) &\leq \text{Lip}(\xi_s) \|R|_{B_{3s}^E(0)}\|_\infty + \|\xi_s\|_\infty \text{Lip}(R|_{B_{3s}^E(0)}) \\ &\leq \frac{1}{s} \text{Lip}(\eta) 3s \text{Lip}(R|_{B_{3s}^E(0)}) + \text{Lip}(R|_{B_{3s}^E(0)}) \\ &= (1 + 3 \text{Lip}(\eta)) \text{Lip}(R|_{B_{3s}^E(0)}) \end{aligned}$$

(using Lemma 2.7 for the first inequality). If $y \in E \setminus B_{3s}^E(0)$ and $z \in E$, then $\|R_s(z) - R_s(y)\| \neq 0$ implies $z \in B_{2s}^E(0)$. In this case, $\|z - y\| \geq s$ and therefore $\|R_s(z) - R_s(y)\| = \|R_s(z)\| \leq \|R(z)\| \leq \text{Lip}(R|_{B_{3s}^E(0)})\|z\| \leq \text{Lip}(R|_{B_{3s}^E(0)})2s \leq \text{Lip}(R|_{B_{3s}^E(0)})2\|z - y\| \leq (1 + 3 \text{Lip}(\eta)) \text{Lip}(R|_{B_{3s}^E(0)})\|z - y\|.$

6.3 Returning to general \mathbb{K} , given arbitrary $\alpha \in]0, 1[$ and $\varepsilon > 0$ we choose $\delta > 0$ so small that (39), (40) and (41) are satisfied.

6.4 In the ultrametric case, we use (7) to find $s \in]0, r]$ such that

$$\text{Lip}(R|_{B_s^E(0)}) \leq \delta \quad (54)$$

and $s \leq 1$. Then $\|R_s(y)\| \leq \text{Lip}(R_s(y))\|y\| \leq \delta s \leq \delta$ whenever $\|R_s(y)\| \neq 0$, and hence

$$\|R_s\|_\infty \leq \delta. \quad (55)$$

6.5 In the real case, (7) provides $s \in]0, r]$ such that

$$\text{Lip}(R|_{B_{3s}^E(0)}) \leq \frac{\delta}{1 + 3 \text{Lip}(\eta)} \quad (56)$$

and $3s \leq 1$. Then again (55) holds.

6.6 Now set $g := R_s$ as just selected, and $h := 0$. Because $\text{Lip}(g) \leq \delta$ by choice of s and $\|g\|_\infty \leq \delta$ by (55), condition (42) is satisfied. Hence both (g, h) and (h, g) satisfy the conditions (16), (17), (30) and (31), by Lemma 4.3. Hence there are unique $v, w \in BC(E, E)$ to which all conclusions of Lemmas 3.1, 3.2 and 4.2 apply. In particular, $v, w \in BL_\alpha(E, E)$ with

$$\text{Lip}_\alpha(v), \text{Lip}_\alpha(w) \leq \varepsilon, \quad (57)$$

and $\text{id}_E + v$ is a homeomorphism with inverse $\text{id}_E + w$. Since $h(0) = 0$ and $g(0) = R(0) = 0$, we also have $v(0) = 0$ and $w(0) = 0$.

6.7 If we are only interested in a single given function f , we can now complete the proof by setting $V := B_s^E(0)$, $U := (\text{id}_E + v)^{-1}(B_s^E(0))$ and $H := (\text{id}_E + v)|_U: U \rightarrow V$. Since

$$(A + g) \circ (\text{id}_E + v) = (\text{id}_E + v) \circ A \quad (58)$$

and $R|_V = g|_V$, we then have

$$f \circ H = f|_V \circ H = (A + g)|_V \circ (\text{id}_E + v)|_U = (\text{id}_E + v) \circ A|_U,$$

from which (8) follows. This completes the proof.

6.8 Since our previous choice of U depends on v (and hence on f), it is unsuitable for the study of parameter dependence. To enable the latter, we need to make a different (usually smaller) choice of U , which we now describe. It is helpful to observe that

$$\tau: [0, \infty[\rightarrow [0, \infty[, \quad \tau(a) := a + \varepsilon a^\alpha \quad (59)$$

is a monotonically increasing bijection, such that $\tau(a) \geq a$ (and hence $\tau^{-1}(a) \leq a$) for all $a \geq 0$. Now

$$(\text{id}_E + v)^{-1}(B_t^E(0)) \supseteq B_{\tau^{-1}(t)}^E(0) \quad \text{for all } t > 0. \quad (60)$$

In fact, given $a > 0$, we have $\|y + v(y)\| \leq \|v\| + \text{Lip}_\alpha(v)\|y\|^\alpha \leq a + \varepsilon a^\alpha = \tau(a)$ for each $y \in B_a^E(0)$, and thus

$$(\text{id}_E + v)(B_a^E(0)) \subseteq B_{\tau(a)}^E(0).$$

Hence $B_a^E(0) \subseteq (\text{id}_E + v)^{-1}(B_{\tau(a)}^E(0))$, entailing (60) (with $a := \tau^{-1}(t)$).

We now set $U := B_{\tau^{-1}(s)}^E(0) \subseteq B_s^E(0)$. Since $V := (\text{id}_E + v)(U) \subseteq B_s^E(0)$ by the preceding discussion, we can set $H := (\text{id}_E + v)|_U: U \rightarrow V$ and complete the discussion as in 6.7.

7 Parameter dependence of the conjugacy

Before we can study parameter dependence of the conjugacies constructed earlier, we compile various auxiliary results. The first lemma is probably part of the folklore. See [18, Theorem 21] for the Lipschitz case; for completeness, the general proof is given in Appendix B.

Lemma 7.1 (Hölder dependence of fixed points on parameters) *Let (X, d_X) and (Y, d_Y) be metric spaces, $\alpha > 0$ and $f: X \times Y \rightarrow Y$ be a mapping with the following three properties:*

- (a) *The family $(f^y)_{y \in Y}$ of the maps $f^y: X \rightarrow Y$, $f^y(x) := f(x, y)$ is uniformly Hölder of exponent α , in the sense that each f^y is Hölder of exponent α and*

$$\mu := \sup\{\text{Lip}_\alpha(f^y): y \in Y\} < \infty.$$

- (b) *The maps $f_x: Y \rightarrow Y$, $y \mapsto f(x, y)$, with $x \in X$, form a uniform family $(f_x)_{x \in X}$ of contractions, in the sense that each f_x is a contraction and*

$$\lambda := \sup\{\text{Lip}(f_x): x \in X\} < 1.$$

- (c) *For each $x \in X$, there exists a fixed point $y_x \in Y$ for f_x .*

Then y_x is uniquely determined and the map $\phi: X \rightarrow Y$, $\phi(x) := y_x$ is Hölder of exponent α , with

$$\text{Lip}_\alpha(\phi) \leq \frac{\mu}{1 - \lambda}.$$

Remark 7.2 Note that condition (a) of Lemma 7.1 is satisfied in particular if f is Hölder of exponent α with respect to some metric d on $X \times Y$ such that $d((x_1, y), (x_2, y)) = d_X(x_1, x_2)$ for all $x_1, x_2 \in X$ and $y \in Y$. Condition (c) is satisfied whenever the metric space (Y, d_Y) is complete (and $Y \neq \emptyset$), by Banach's Fixed Point Theorem.

The dependence of w on v in the situation of Lemma 2.8 is considered next.

Lemma 7.3 *Let $(E, \|\cdot\|)$ be a Banach space over a valued field $(\mathbb{K}, |\cdot|)$ (such that $E \neq \{0\}$), and $0 < \lambda < 1$. Let $A: E \rightarrow E$ be an automorphism of topological vector spaces, and Ω be the set of all bounded, Lipschitz maps $v: E \rightarrow E$ such that*

$$\text{Lip}(v)\|A^{-1}\| \leq \lambda.$$

Equip Ω with the metric given by $d_\infty(v_1, v_2) := \|v_1 - v_2\|_\infty$. Given $v \in \Omega$, let

$$w_v := (A + v)^{-1} - A^{-1}.$$

Then the map $\phi: \Omega \rightarrow BC(E, E)$, $v \mapsto w_v$ is Lipschitz, with

$$\text{Lip}(\phi) \leq \frac{\|A^{-1}\|}{1 - \lambda}.$$

Proof. Consider the map

$$h: \Omega \times BC(E, E) \rightarrow BC(E, E), \quad h(v, u) := -A^{-1} \circ v \circ (A^{-1} + u).$$

We know from (15) that w_v satisfies

$$w_v = -A^{-1} \circ v \circ (A^{-1} + w_v).$$

Thus w_v is a fixed point of $h_v := h(v, \cdot)$, and it only remains to verify the hypotheses of Lemma 7.1 for h , with $\mu \leq \|A^{-1}\|$ and the given λ . Each h_v is Lipschitz, with $\text{Lip}(h_v) \leq \|A^{-1}\| \text{Lip}(v) \leq \lambda$. Hence $(h_v)_{v \in \Omega}$ is a uniform family of contractions. Fix $u \in BC(E, E)$. Given $v_1, v_2 \in \Omega$, we have

$$\|h(v_2, u) - h(v_1, u)\|_\infty = \|A^{-1} \circ (v_2 - v_1) \circ (A^{-1} + u)\|_\infty \leq \|A^{-1}\| \|v_2 - v_1\|_\infty.$$

Hence $h(\cdot, u): \Omega \rightarrow BC(E, E)$ is Lipschitz with $\text{Lip}(h(\cdot, u)) \leq \|A^{-1}\|$, which completes the proof. \square

A linear map $A: E \rightarrow F$ between Banach spaces over a locally compact, valued field is called a *compact operator* if $A(B)$ is relatively compact in F for each bounded subset $B \subseteq E$ (or equivalently, if $A(B_1^E(0)) \subseteq F$ is relatively compact). Then A is continuous. As it is similar to the classical real case, we relegate the proof of the next result to the appendix (Appendix B).

Lemma 7.4 *Let (K, d) be a compact metric space, $(E, \|\cdot\|)$ be a normed space over a valued field $(\mathbb{K}, |\cdot|)$, and $\alpha > \beta > 0$. Then $L_\alpha(K, E) \subseteq L_\beta(K, E)$.*

Assume that, moreover, \mathbb{K} is locally compact and E of finite dimension. If $|\cdot|$ is ultrametric, assume also that d is ultrametric. Then the inclusion map

$$j_{\beta,\alpha}: L_\alpha(K, E) \rightarrow L_\beta(K, E), \quad f \mapsto f$$

is a compact operator.

Lemma 7.5 *Let (K, d) be a compact metric space and $X \subseteq K$ be a dense subset. Let $(E, \|\cdot\|)$ be a finite-dimensional normed space over a valued field $(\mathbb{K}, |\cdot|)$ that is locally compact, and $\alpha > \beta > 0$. If $|\cdot|$ is ultrametric, also assume that also d is ultrametric. Let $B \subseteq BL_\alpha(X, E)$ be bounded; thus*

$$\sup_{f \in B} \|f\|_\infty < \infty \quad \text{and} \quad \sup_{f \in B} \text{Lip}_\alpha(f) < \infty.$$

Then $BC(X, E)$ and $BL_\beta(X, E)$ induce the same topology on B .

Proof. Assume first that $X = K$. By Lemma 7.4, the closure $\overline{B} \subseteq L_\beta(K, E)$ is compact. Because the topology on \overline{B} induced by $C(K, E)$ is Hausdorff and coarser than the previous compact topology, the two topologies coincide. The same then holds for the topologies on the smaller set B . In the general case, each $f \in BL_\alpha(X, E)$ extends (by uniform continuity) uniquely to a continuous function $\tilde{f}: K \rightarrow E$. Then $\text{Lip}_\alpha(f) = \text{Lip}_\alpha(\tilde{f})$ (as we can pass to limits in (x, y) in the Hölder condition), and thus $BL_\alpha(X, E) \rightarrow BL_\alpha(K, E)$, $f \mapsto \tilde{f}$ is an isometric isomorphism. Likewise with β in place of α . The assertion hence follows from the result for maps on K , as just proved. \square

Theorem 7.6 *Let E be a Banach space over a valued field $(\mathbb{K}, |\cdot|)$ and $d_\infty: BC(E, E)^2 \rightarrow [0, \infty[$, $d_\infty(h_1, h_2) := \|h_1 - h_2\|_\infty$ be the supremum metric. Let $A: E \rightarrow E$ be a hyperbolic automorphism and $\alpha \in]0, 1[$ as well as $\varepsilon, \delta > 0$ be such that (39)–(41) from Lemma 4.3 are satisfied. Let Ω be the set of all bounded, Lipschitz maps $g: E \rightarrow E$ such that $\max\{\|g\|_\infty, \text{Lip}(g)\} \leq \delta$. For $g \in \Omega$, let $v_g, w_g: E \rightarrow E$ be the bounded continuous maps determined by*

$$(A + g) \circ (\text{id}_E + v_g) = (\text{id}_E + v_g) \circ A$$

and $w_g := (\text{id}_E + v_g)^{-1} - \text{id}_E$. Set $\sigma(g) := v_g$, $\tau(g) := w_g$. Then σ is Lipschitz as map from (Ω, d_∞) to $(BC(E, E), d_\infty)$, and $\tau: (\Omega, d_\infty) \rightarrow (BC(E, E), d_\infty)$ is Hölder of exponent α . If \mathbb{K} is locally compact and E finite-dimensional, then σ and τ are also continuous as maps from (Ω, d_∞) to $(BL_\beta(E, E), \|\cdot\|_\beta)$, for each $\beta < \alpha$.

Proof. Throughout the proof, we equip $BC(E, E)$ and Ω with the supremum metric d_∞ . Moreover, we give $\Omega \times BC(E, E)$ the metric d defined via $d((g_1, v_1), (g_2, v_2)) := \max\{d_\infty(g_1, g_2), d_\infty(v_1, v_2)\}$. Given $g \in \Omega$, define $f(g, v) := \theta(v) = (\theta_1(v), \theta_2(v))$ for $v \in BC(E, E)$ as in (22) and (23) (applied with $h := 0$). We claim that

$$f: (\Omega \times BC(E, E), d) \rightarrow (BC(E, E), d_\infty)$$

satisfies the hypotheses of the Lipschitz case of Lemma 7.1. If this is true, then the map $\sigma: \Omega \rightarrow BC(E, E)$ taking $g \in \Omega$ to the fixed point $\sigma(g) := v_g$ of $f_g := f(g, \cdot): BC(E, E) \rightarrow BC(E, E)$ is Lipschitz. To establish the claim, note first that condition (c) of Lemma 7.1 is satisfied by completeness of $BC(E, E)$ (see Remark 7.2). Condition (b) is satisfied since (26) and (17) show that

$$\text{Lip}(f_g) \leq \max\{\|A_2^{-1}\|(1 + \delta), \|A_1\| + \delta\},$$

where the right hand side is < 1 and independent of $g \in \Omega$. To see that the maps $f^v := f(\cdot, v): \Omega \rightarrow BC(E, E)$, for $v \in BC(E, E)$, are uniformly Lipschitz, note that

$$f^v(g) - f^v(k) = ((g_s - k_s) \circ (\text{id}_E + v) \circ A^{-1}, A_2^{-1} \circ (k_u - g_u) \circ (\text{id}_E + v))$$

for $g, k \in BC(E, E)$ and thus

$$\begin{aligned} d_\infty(f^v(g), f^v(k)) &\leq \max\{\|g_s - k_s\|_\infty, \|A_2^{-1}\| \|k_u - g_u\|\} \\ &\leq \max\{1, \|A_2^{-1}\|\} d_\infty(k, g). \end{aligned}$$

Hence $\text{Lip}(f^v) \leq \max\{1, \|A_2^{-1}\|\}$, for all $v \in BC(E, E)$.

Now define Y as in (32). For fixed $h \in \Omega$ and $g := 0$, let $\theta = (\theta_1, \theta_2)$ be as in (22) and (23), and recall from the proof of Lemma 4.2 that θ restricts to a contraction $f_h := \theta|_Y^Y$ of Y . To see that τ is Hölder, we need only show that the map $f: \Omega \times Y \rightarrow Y$, $f(h, x) := f_h(x)$ satisfies the hypotheses of Lemma 7.1 (using the metric d_∞ on Y and d on the left hand side). By the proof of Lemma 4.2, Y is complete with respect to d_∞ . Thus condition (c) of Lemma 7.1 is satisfied, and (b) can be shown as in the first part of this proof. To verify (a), let $v \in Y$. For $h, k \in \Omega$, the first and second components of $f^v(h) - f^v(k)$ are given by

$$A_1 \circ (v_s \circ (A + h)^{-1} - v_s \circ (A + k)^{-1}) + k_s \circ (A + k)^{-1} - h_s \circ (A + h)^{-1} \text{ and } (61)$$

$$A_2^{-1} \circ (h_u - k_u) + A_2^{-1} \circ (v_u \circ (A + h)^{-1} - v_u \circ (A + k)^{-1}), \quad (62)$$

respectively. The supremum norm of (61) is bounded by

$$\begin{aligned} & \|A_1\| \operatorname{Lip}_\alpha(v_s) \|(A + h)^{-1} - (A + k)^{-1}\|_\infty^\alpha + \|k_s - h_s\|_\infty \\ & + \operatorname{Lip}_\alpha(h_s) \|(A + k)^{-1} - (A + h)^{-1}\|_\infty^\alpha, \end{aligned} \quad (63)$$

where $\operatorname{Lip}_\alpha(h_s) \leq \max\{\operatorname{Lip}(h_s), 2\|h_s\|_\infty\} \leq 2\delta$ by Lemma 2.5, $\|k_s - h_s\|_\infty \leq \rho\|k_s - h_s\|_\infty^\alpha$ with $\rho := \max\{1, 2\delta\}$ and

$$\|(A + k)^{-1} - (A + h)^{-1}\|_\infty \leq \frac{\|A^{-1}\|}{1 - \delta\|A^{-1}\|} \|k - h\|_\infty$$

by Lemma 7.3. Hence the following is an upper bound for (63):

$$(\|A_1\|\varepsilon + 2\delta) \left(\frac{\|A^{-1}\|}{(1 - \delta\|A^{-1}\|)} \right)^\alpha \|k - h\|_\infty^\alpha + \rho\|k - h\|_\infty^\alpha. \quad (64)$$

Likewise, the supremum norm of (62) is bounded by

$$\|A_2^{-1}\|\rho\|h - k\|_\infty^\alpha + \|A_2^{-1}\| \underbrace{\operatorname{Lip}_\alpha(v_u)}_{\leq \varepsilon} \left(\frac{\|A^{-1}\|}{(1 - \delta\|A^{-1}\|)} \right)^\alpha \|k - h\|_\infty^\alpha. \quad (65)$$

Taking now the maximum of the bounds provided by (64) and (65), we see that $\|f^v(h) - f^v(k)\|_\infty \leq M\|h - k\|_\infty^\alpha$ for $h, k \in \Omega$, with some constant M independent of v , h , and k . \square

A Existence of ultrametric adapted norms

Lemma A.1 *Let $(E, \|\cdot\|)$ be an ultrametric Banach space over a valued field $(\mathbb{K}, |\cdot|)$, and $A: E \rightarrow E$ be a hyperbolic automorphism. Then there exists an ultrametric norm $\|\cdot\|^\sim$ on E adapted to E .*

Proof. We first assume that $E = E_s$; without loss of generality $E \neq \{0\}$. Let $\|\cdot\|'$ be a (not necessarily ultrametric) norm on E adapted to A . Since the norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent, there exists $C \geq 1$ such that $C^{-1}\|\cdot\|' \leq \|\cdot\| \leq C\|\cdot\|'$. Let $\theta := \|A\|' < 1$ be the operator norm of A with respect to

$\|\cdot\|'$. Choose an integer $n \geq 2$ so large that $\sigma := C^2\theta^{n-1} < 1$ and define an ultrametric norm $\|\cdot\|^\sim$ on E equivalent to $\|\cdot\|$ via

$$\|x\|^\sim := \max\{\theta^{-\frac{k}{n-1}}\|A^k x\| : k = 0, \dots, n-1\}.$$

The operator norm $\|A^n\|$ of A^n with respect to $\|\cdot\|$ satisfies $\|A^n\| \leq C^2\|A^n\|' \leq C^2(\|A\|')^n = C^2\theta^n$. To see that $\|\cdot\|^\sim$ is adapted, let $x \in E$. Then $\|Ax\|^\sim$ is the maximum of $\max\{\theta^{\frac{1-k}{n-1}}\|A^k x\| : k = 1, \dots, n-1\} \leq \theta^{\frac{1}{n-1}}\|x\|^\sim$ and $\theta^{-\frac{n-1}{n-1}}\|A^n x\| \leq C^2\theta^{n-1}\|x\| \leq C^2\theta^{n-1}\|x\|^\sim$. By the preceding, the operator norm $\|A\|^\sim$ of A with respect to $\|\cdot\|^\sim$ satisfies

$$\|A\|^\sim \leq \max\{\theta^{\frac{1}{n-1}}, \sigma\} < 1.$$

Hence $\|\cdot\|^\sim$ is an adapted norm on $E = E_s$.

In a general case, $E = E_s \oplus E_u$, the preceding arguments provide ultrametric norms $\|\cdot\|_1$ on E_s adapted to $A|_{E_s}$ and $\|\cdot\|_2$ on E_u adapted to $A^{-1}|_{E_u}$. Then $\|x + y\|^\sim := \max\{\|x\|_1, \|y\|_2\}$ for $x \in E_s, y \in E_u$ defines an ultrametric norm on E adapted to A . \square

B Proof of Lemma 7.1

If also z_x is a fixed point of f_x , then $d_Y(y_x, z_x) = d_Y(f_x(y_x), f_x(z_x)) \leq \lambda d_Y(y_x, z_x)$, whence $d_Y(y_x, z_x) = 0$ and hence $z_x = y_x$. For $v \in X$ and $y \in Y$, we have $f_v^n(y) \rightarrow y_v$ as $n \rightarrow \infty$ since $d_Y(f_v^n(y), y_v) = d_Y(f_v^n(y), f_v^n(y_v)) \leq \lambda^n d_Y(y, y_v)$. In particular, $f_v^n(y_w) \rightarrow y_v$ for each $w \in X$. We claim:

$$d_Y(f_v^n(y_w), y_w) \leq \mu d_X(v, w)^\alpha \sum_{k=0}^{n-1} \lambda^k \quad \text{for all } n \in \mathbb{N}.$$

If this is true, letting $n \rightarrow \infty$ we deduce that

$$d_Y(y_v, y_w) \leq \mu d_X(w, v)^\alpha \sum_{k=0}^{\infty} \lambda^k = \frac{\mu}{1-\lambda} d_X(w, v)^\alpha,$$

as required. If $n = 1$, we have $d_Y(f_v(y_w), y_w) = d_Y(f_v(y_w), f_w(y_w)) = d_Y(f^{y_w}(v), f^{y_w}(w)) \leq \mu d_X(v, w)^\alpha$, verifying the claim in this case. Assuming

that the claim is true for some n , we obtain

$$\begin{aligned}
d_Y(f_v^{n+1}(y_w), y_w) &= d_Y(f_v^{n+1}(y_w), f_w(y_w)) \\
&\leq d_Y(f_v(f_v^n(y_w)), f_v(y_w)) + d_Y(f_v(y_w), f_w(y_w)) \\
&\leq \lambda d_Y(f_v^n(y_w), y_w) + \mu d_X(v, w)^\alpha \\
&\leq \lambda \mu d_X(v, w)^\alpha \sum_{k=0}^{n-1} \lambda^k + \mu d_X(v, w)^\alpha \\
&= \mu d_X(v, w)^\alpha \sum_{k=0}^n \lambda^k,
\end{aligned}$$

as required. This induction proves the claim.

C Proof of Lemma 7.4

The first assertion is covered by Lemma 2.5. Now assume that \mathbb{K} is locally compact (whence \mathbb{K} is \mathbb{R} or \mathbb{C} as a topological field in the archimedean case), and assume that E is finite-dimensional. Then $E \cong \mathbb{K}^n$ (equipped with product topology) for some $n \in \mathbb{N}_0$ as a topological vector space (see Theorem 2 in [5, Chapter I, §2, no. 3]), whence E is locally compact.

In the real or complex case, define $a := 1$ and $\zeta:]0, \infty[\rightarrow]0, \infty[, \zeta(t) := t$. If $|\cdot|$ and d are ultrametric, let $a \in \mathbb{K}$ with $0 < |a| < 1$. Define $\zeta:]0, \infty[\rightarrow \mathbb{K}$ via

$$\zeta(t) := a^k \quad \text{if } k \in \mathbb{Z} \text{ and } |a|^{k+1} < t \leq |a|^k. \quad (66)$$

Thus, in either case,

$$|a| \cdot |\zeta(t)| < t \leq |\zeta(t)| \quad \text{for all } t > 0. \quad (67)$$

Let $D := \{(x, y) \in K \times K : x \neq y\}$ and consider the map

$$D \rightarrow \mathbb{K}, \quad (x, y) \mapsto \zeta(d(x, y)^\beta).$$

The continuity of this map is obvious in the real and complex cases. In the ultrametric case, continuity follows from the fact that

$$\{(x, y) \in K \times K : d(x, y) = t\}$$

is open in $K \times K$ for each $t > 0$ (cf. (9)).

We equip $C(K, E)$ with $\|\cdot\|_\infty$, let $\phi_1: L_\beta(K, E) \rightarrow C(K, E)$ be the inclusion map, and define

$$\phi_2: L_\beta(K, E) \rightarrow BC(D, E)$$

via $\phi_2(f)(x, y) := \frac{f(y) - f(x)}{\zeta(d(y, x)^\beta)}$. As a consequence of (67),

$$\|\phi_2(f)\|_\infty \leq \text{Lip}_\beta(f) \leq |a|^{-1} \|\phi_2(f)\|_\infty \quad \text{for each } f \in L_\beta(K, E), \text{ whence}$$

$$\phi = (\phi_1, \phi_2): L_\beta(K, F) \rightarrow C(K, E) \times BC(D, E)$$

is a topological embedding. Moreover ϕ has closed image. To see this, suppose that $\phi(f_n) \rightarrow (f, g)$ as $n \rightarrow \infty$. Then

$$g(x, y) = \lim_{n \rightarrow \infty} \phi_2(f_n)(x, y) = \lim_{n \rightarrow \infty} \frac{f_n(y) - f_n(x)}{\zeta(d(y, x)^\beta)} = \frac{f(y) - f(x)}{\zeta(d(y, x)^\beta)},$$

entailing that f is Hölder with $\text{Lip}_\beta(f) \leq |a|^{-1} \|g\|_\infty$, and $g = \phi_2(f)$. Thus $(f, g) = \phi(f)$.

Now abbreviate $B := \{f \in L_\alpha(K, E): \|f\|_\alpha < 1\}$, and let \overline{B} be the closure of B in $C(K, E)$. Then

$$\text{Lip}_\beta(f) \leq \max\{\text{Lip}_\alpha(f), 2\|f\|_\infty\} \leq 2 \quad \text{for all } f \in B,$$

using (2.5). Given $x \in K$ and $\varepsilon > 0$, let $\delta := \varepsilon^{\frac{1}{\alpha}}$. For each $y \in B_\delta^K(x)$ and $f \in B$, we then have $\|f(y) - f(x)\| \leq \text{Lip}_\alpha(f) d(x, y)^\alpha \leq \delta^\alpha = \varepsilon$. Thus B is equicontinuous. Since, moreover, $\{f(x): x \in B\} \subseteq \overline{B}_1^E(0)$ is relatively compact for each $x \in K$, Ascoli's Theorem shows that $\overline{B} \subseteq C(K, E)$ is compact. We claim that also $\overline{\phi_2(B)} \subseteq BC(D, E)$ is compact. If this is true, then $C := \text{im}(\phi) \cap (\overline{B} \times \overline{\phi_2(B)})$ is compact and hence also $\phi^{-1}(C)$ is compact. Since $B \subseteq \phi^{-1}(C)$, this proves the lemma.

To verify the claim, let $\varepsilon > 0$ be given. We can choose $\sigma > 0$ so small that

$$2\sigma^{\alpha-\beta} \leq \varepsilon. \tag{68}$$

We let D_σ be the set of all $(x, y) \in K \times K$ such that $\frac{\sigma}{9} \leq d(x, y) \leq 2$. Since D_σ is compact, the continuous map $\gamma: D_\sigma \rightarrow \mathbb{K}$, $(x, y) \mapsto \frac{1}{\zeta(d(x, y)^\beta)}$ is uniformly continuous. Hence, there exists $\delta > 0$ such that

$$|\gamma(x, y) - \gamma(x', y')| \leq \varepsilon/3$$

for all $(x, y), (x', y') \in D_\sigma$ such that $d(x, x') < \delta$ and $d(y, y') < \delta$. After shrinking δ if necessary, we may assume that also

$$\delta \leq \sigma/9 \quad \text{and} \quad \frac{2\delta^\alpha}{(\sigma/3)^\beta} \leq \varepsilon/3. \quad (69)$$

Let $(x, y), (x', y') \in D$ with $d(x, x') < \delta$ and $d(y, y') < \delta$. We show that

$$\|\phi_2(f)(x', y') - \phi_2(f)(x, y)\| \leq \varepsilon, \quad (70)$$

for all $f \in B$. If this is true, then the function $\phi_2(f)$ is uniformly continuous and hence has a unique continuous extension $\psi(f): \overline{D} \rightarrow E$ to the compact closure $\overline{D} \subseteq K \times K$. Letting (x, y) and (x', y') as before pass to limits in \overline{D} , we deduce from (70) that also

$$\|\psi(f)(x', y') - \psi(f)(x, y)\| \leq \varepsilon,$$

for all $f \in B$, $(x, y) \in \overline{D}$ and $(x', y') \in \overline{D}$ such that $d(x, x') < \delta$ and $d(y, y') < \delta$. Hence $\Omega := \{\psi(f): f \in B\}$ is an equicontinuous set of functions in $C(\overline{D}, E)$. Given $(x, y) \in D$, we have $\|\psi(f)(x, y)\| \leq \text{Lip}_\beta(f) \leq 2$ for each $f \in B$ (and, by continuity, this then also holds for all $(x, y) \in \overline{D}$). Hence $\{\psi(f)(x, y): f \in B\} \subseteq \overline{B}_2^E(0)$ and thus the equicontinuous set Ω is also pointwise relatively compact. Hence, by Ascoli's Theorem, Ω is relatively compact in $C(\overline{D}, E)$. Because the restriction map

$$C(\overline{D}, E) \rightarrow BC(D, E) \quad h \mapsto h|_D$$

is continuous linear and takes Ω to B , we deduce that also B is relatively compact, as claimed.

It only remains to verify (70). There are two cases. If $d(y, x) < \sigma/3$, then $d(y', x') \leq \sigma$ (as we assume that $d(x', x), d(y', y) < \delta \leq \sigma/9$) and hence

$$\begin{aligned} \|\phi_2(f)(x', y') - \phi_2(f)(x, y)\| &\leq \|\phi_2(f)(x', y')\| + \|\phi_2(f)(x, y)\| \\ &\leq \frac{\|f(y') - f(x')\|}{d(y', x')^\beta} + \frac{\|f(y) - f(x)\|}{d(y, x)^\beta} \\ &\leq \text{Lip}_\alpha(f)(d(y', x')^{\alpha-\beta} + d(y, x)^{\alpha-\beta}) \\ &\leq 2\sigma^{\alpha-\beta} \leq \varepsilon, \end{aligned}$$

by (68). If $d(y, x) \geq \sigma/3$, then $d(y', x') \geq d(y, x) - d(y', y) - d(x', x) \geq \frac{\sigma}{9}$ and

$$\begin{aligned}
& \|\phi_2(f)(x', y') - \phi_2(f)(x, y)\| \\
& \leq \frac{\|f(y) - f(x) - f(y') + f(x')\|}{|\zeta(d(y, x)^\beta)|} \\
& \quad + \underbrace{\left| \frac{1}{\zeta(d(y', x')^\beta)} - \frac{1}{\zeta(d(y, x)^\beta)} \right|}_{\leq \varepsilon/3} \underbrace{\|f(y') - f(x')\|}_{\leq 2} \\
& \leq \frac{1}{d(y, x)^\beta} \text{Lip}_\alpha(f)(d(y, y')^\alpha + d(x, x')^\alpha) + 2\varepsilon/3 \\
& \leq \frac{2\delta^\alpha}{(\sigma/3)^\beta} + 2\varepsilon/3 \leq \varepsilon,
\end{aligned}$$

using (69) for the final inequality.

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